Multiple solutions for a quasilinear Schrödinger equation on $\mathbb{R}^N$

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Abstract

The multiplicity of positive weak solutions for a quasilinear Schrödinger equations $-L_pu + (\lambda A(x) + 1)|u|^{p-2}u = h(u)$ in $\mathbb{R}^N$ is established, where $L_pu \doteq \epsilon^p\Delta_pu + \epsilon^p\Delta_p(u^2)u$, A is a nonnegative continuous function and nonlinear term h has a subcritical growth. We achieved our results by using minimax methods and Lusternik-Schnirelman theory of critical points.

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1 Introduction

In this paper we establish existence and multiplicity of positive weak solutions for the following class of quasilinear Schrödinger equations:

$$-L_p u + (\lambda A(x) + 1)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \qquad (P_{\epsilon,\lambda})$$

where

$$L_p u \doteq \epsilon^p \Delta_p u + \epsilon^p \Delta_p(u^2)u,$$

$\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, ϵ, λ are positive parameters, $2 \leq p < N$ and function $A : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(A_1) $A \in C^1(\mathbb{R}^N, \mathbb{R}), A(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\Omega = \text{int}A^{-1}(0)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$ and $0 \in \Omega$. Moreover, $A^{-1}(0) = \overline{\Omega} \cup D$ where D is a set of measure zero.

(A_2) There exists $K_0 > 0$ such that

$$\mu\Big(\Big\{x \in \mathbb{R}^N : A(x) \leq K_0\Big\}\Big) < \infty,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^N$.

On the nonlinearity h, we assume that it is of class C^1 and satisfies the following conditions:

(H_1) $h'(s) = o(|s|^{p-2})$ at the origin;

(H_2) $\lim\limits_{|s|\to\infty} h'(s)|s|^{-q+2} = 0$ for some $q \in (2p, 2p^*)$ where $p^* = Np/(N-p)$;

(H_3) There exists $\theta > 2p$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$.

(H_4) The function $s \to h(s)/s^{2p-1}$ is increasing for $s > 0$.

A typical example of a function satisfying the conditions $(H_1) - (H_4)$ is given by $h(s) = s^\mu$ for $s \geq 0$, with $2p - 1 < \mu < q - 1$, and $h(s) = 0$ for $s < 0$.

For $p = 2$, the solutions of $(P_{\epsilon,\lambda})$ are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t\psi = -\Delta\psi + V(x)\psi - \widetilde{h}(|\psi|^2)\psi - \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \qquad (1.1)$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, V is a given potential, κ is a real constant and $\rho, \widetilde{h}$ are real functions. Quasilinear equations of the form (1.1) have been studied in relation with some mathematical models in physics. For example, when $\rho(s) = s$, the above equation is

$$i\partial_t \psi = -\Delta \psi + V(x)\psi - \kappa \Delta[|\psi|^2]\psi - \widetilde{h}(|\psi|^2)\psi. \tag{1.2}$$

It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.2) and numerics results on this equation are obtained in [9]. In [16], motivated by the nanotubes and fullerene related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.2) and numerics results have been done on domains of disc type, cylinder type and sphere type. The superfluid film equation in plasma physics has also the structure (1.1) for $\rho(s) = s$, see [19].

The general equation (1.1) with various form of quasilinear terms $\rho(s)$ has been derived as models of several other physical phenomena corresponding to various types of $\rho(s)$. For example, in the case $\rho(s) = (1 + s)^{1/2}$, equation (1.1) models the self-channeling of a high-power ultra short laser in matter, see [10] and [26]. Equation (1.1) also appears in fluid mechanics [18], in the theory of Heisenberg ferromagnets and magnons [31], in dissipative quantum mechanics and in condensed matter theory [23]. The Semilinear case corresponding to $\kappa = 0$ in the whole $\mathbb{R}^N$ has been studied extensively in recent years, see for example [15], [17] and references therein.

Putting $\psi(t, x) = \exp(-iFt)u(x)$, $F \in \mathbb{R}$, into the equation (1.2), we obtain a corresponding equation

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u) \tag{1.3}$$

where we have renamed $V(x) - F$ to be $V(x)$, $h(u) = \widetilde{h}(u^2)u$ and we assume, without loss of generality, that $\kappa = 1$.

The quasilinear equation (1.3) in the whole $\mathbb{R}^N$ has received special attention in the past several years, see for example the works [1], [4], [5], [6], [11], [12], [13], [14], [20], [21], [22], [24], [29], [30] and references therein. In these papers, we find important results on the existence of nontrivial solutions of (1.3) and a good insight into this quasilinear Schrödinger equation. The main strategies used are the following: the first of them consists in by using a constrained minimization argument, which gives a solution of (1.3) with an

unknown Lagrange multiplier λ in front of the nonlinear term, see for example [24]. The other one consists in by using a special change of variables to get a new semilinear equation and an appropriate Orlicz space framework, for more details see [11], [13] and [21]. In [1], existence, multiplicity and concentration of solutions have been study, by using the Lusternik-Schnirelman category, for the following class of problems

$$-\epsilon^p \Delta_p u - \epsilon^p \Delta_p (u^2) u + V(x)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N),$$

when ϵ is sufficiently small and V satisfying the condition

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0, \tag{R}$$

which has been introduced by Rabinowitz [25]. In [2], multiplicity of solutions also have been proved for problem of the type

$$-\epsilon^p \Delta_p u - \epsilon^p \Delta_p (u^2) u = h(u), \quad u \in W_0^{1,p}(\lambda\Omega),$$

where Ω is a bounded domain and λ is large enough.

Related to the p-Laplacian operator, we would like to cite a paper due to Alves and Soares [3], where existence, multiple and concentration of solutions, by using the Lusternik-Schnirelman category, have been established for the following class of p-Laplacian equations

$$-\epsilon^p \Delta_p u + (\lambda A(x) + 1)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \tag{P}$$

by assuming that A verifies conditions $(A_1)-(A_2)$, h is a continuous functions with subcritical growth, ϵ is sufficiently small and λ is large enough. In that paper, it is proved that there exists $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$ there exists $\lambda^*(\epsilon) > 0$ such that (P) has at least $cat(\Omega)$ solutions for any $\lambda \geq \lambda^*(\epsilon)$. The results showed in [3] are associated with the main results proved by Barstch and Wang [7, 8], where condition (A_2) was used by the first time. Here, we would like to emphasize that the assumptions $(A_1)-(A_2)$ do not imply that potential $V(x) = \lambda A(x) + 1$ verifies condition (R).

The present paper was motivated by works [1], [2] and [3]. Here, we intend to show that the same type of results found in [3] for p-Laplacian also hold for operator L_p. However, due to the presence of the term $\Delta_p(u^2)u$ in $L_p u$, several estimates used [3] can not be repeated for the functional energy associated to $(P_{\epsilon,\lambda})$, given by

$$J_{\epsilon,\lambda}(u) = \frac{1}{p}\int_{\mathbb{R}^N} \epsilon^p(1 + 2^{p-1}|u|^p)|\nabla u|^p + \frac{1}{p}\int_{\mathbb{R}^N} (\lambda A(x) + 1)|u|^p - \int_{\mathbb{R}^N} H(u),$$

where $H(s) = \int_0^s h(t)dt$. As observed in [27] and [28] , there are some technical difficulties to apply directly variational methods to $J_{\epsilon,\lambda}$. The main difficult is related to the fact that $J_{\epsilon,\lambda}$ is not well defined in $W^{1,p}(\mathbb{R}^N)$. By a direct computation, if $u \in C_0^1(\mathbb{R}^N \backslash \{0\})$ is defined by

$$u(x) = |x|^{(p-N)/2p} \quad \text{for} \quad x \in B_1 \backslash \{0\},$$

then $u \in W^{1,p}(\mathbb{R}^N)$ while the function $|u|^p |\nabla u|^p$ does not belong to $L^1(\mathbb{R}^N)$. To overcome this difficulty, we use a change variable developed in [27] and [28], which generalize one found in Liu, Wang and Wang [21] and Colin-Jeanjean [11] for the case $p = 2$.

Before to state our main result, we recall that if Y is a closed set of a topological space X, we denote the Lusternik-Schnirelman category of Y in X by $cat_X(Y)$, which is the least number of closed and contractible sets in X that cover Y. Hereafter, $cat\, X$ denotes $cat_X(X)$.

The main result that we prove is the following:

Theorem 1.1 *Suppose that* $(A_1) - (A_2)$ *and* $(H_1) - (H_4)$ *hold. Then there exists* $\epsilon^* > 0$ *such that for any* $\epsilon \in (0, \epsilon^*)$ *there exists* $\lambda^*(\epsilon) > 0$ *such that* $(P_{\epsilon,\lambda})$ *has at least* $cat(\Omega)$ *solutions for any* $\lambda \geq \lambda^*(\epsilon)$.

To finish this introduction, we would like to emphasize that Theorem 1.1 can be seen as a complement of the studies made in [1] and [3]. In [1], the authors considered a class of problems involving the L_p operator, however the potential V verifies the condition (R), while that in [3], only the p-Laplacian was considered.

The plan of this paper is as follows. In Section 2, we review some proprieties of the change variable that we will apply. Section 3 establishes a compactness result for the energy functional for all sufficiently large λ and arbitrary ϵ. In Section 4 is made a study of the behavior of the minimax levels with respect to parameter λ and ϵ. Section 5 offers the proof of our main result.

2 Variational framework and preliminary results

Since we intend to find positive solutions, let us assume that

$$h(s) = 0 \quad \text{for all} \quad s < 0.$$

Moreover, hereafter, we will work with the following problem equivalent to $(P_{\epsilon,\lambda})$, which is obtained under change of variable $\epsilon z = x$

$$\begin{cases} -\Delta_p u - \Delta_p(u^2)u + (\lambda A(\epsilon x)+1)|u|^{p-2}u = h(u) \;\; in \;\; \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \;\text{ with }\; 2 \le p < N, \\ u(x) > 0, \forall\, x \in \mathbb{R}^N. \end{cases} \qquad (P^*_{\epsilon,\lambda})$$

In what follows, we use the change variable developed in [27] and [28] which generalizes one found in Liu, Wang and Wang [21] and Colin-Jeanjean [11] for the case $p = 2$. More precisely, let us introduce the change of variables $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1+2^{p-1}|f(t)|^p)^{1/p}} && \text{on } \;[0,+\infty), \\ f(t) &= -f(-t) && \text{on } \;(-\infty,0]. \end{aligned} \tag{2.1}$$

Therefore, using the above change of variables, we consider a new functional $I_{\epsilon,\lambda}$, given by

$$I_{\epsilon,\lambda}(v) \doteq J_{\epsilon,\lambda}(f(v)) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p + \frac{1}{p}\int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(v)|^p - \int_{\mathbb{R}^N} H(f(v)) \tag{2.2}$$

which is well defined on the Banach space X defined by

$$X_{\epsilon,\lambda} = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} A(\epsilon x)|f(v)|^p < \infty \right\}$$

endowed with the norm

$$\|u\|_{\epsilon,\lambda} = |\nabla u|_p + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(\xi v)|^p\right].$$

A direct computation shows that $I_{\epsilon,\lambda} : X_{\epsilon,\lambda} \to \mathbb{R}$ is of class C^1 under the conditions $(A_1)-(A_2)$ and $(H_1)-(H_2)$ with

$$I'_{\epsilon,\lambda}(v)w = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla w \; + (\lambda A(\epsilon x)+1)|f(v)|^{p-2} f(v) f'(v) w -$$

$$- \int_{\mathbb{R}^N} h(f(v)) f'(v) w$$

for $v, w \in W^{1,p}(\mathbb{R}^N)$. Thus, the critical points of $I_{\epsilon,\lambda}$ correspond exactly to the weak solutions of the problem

$$\begin{cases} -\Delta_p v + (\lambda A(\epsilon x) + 1)|f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v) \text{ in } \mathbb{R}^N \\ v \in W^{1,p}(\mathbb{R}^N) \text{ with } 2 \leq p < N, \\ v(x) > 0, \forall\, x \in \mathbb{R}^N. \end{cases} \tag{$S_{\epsilon,\lambda}$}$$

The below result establishes a relation between the solutions of $(S_{\epsilon,\lambda})$ with one of $(P^*_{\epsilon,\lambda})$:

Proposition 2.1 *If $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ is a critical point of the functional $I_{\epsilon,\lambda}$, then $u = f(v)$ is a weak solution of $(P^*_{\epsilon,\lambda})$.*

Proof. See [28]. ∎

From the above proposition, it is clear that to obtain a weak solution of $(P_{\epsilon,\lambda})$, it is sufficient to obtain a critical point of the functional $I_{\epsilon,\lambda}$ in $W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$.

Next, let us collect some properties of the change of variables $f : \mathbb{R} \to \mathbb{R}$ defined in (2.1), which will be usual in the sequel of the paper.

Lemma 2.1 *The function $f(t)$ and its derivative enjoy the following properties:*

(1) f is uniquely defined, C^2 and invertible;

(2) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;

(3) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;

(4) $f(t)/t \to 1$ as $t \to 0$;

(5) $|f(t)| \leq 2^{1/2p}|t|^{1/2}$ for all $t \in \mathbb{R}$;

(6) $f(t)/2 \leq tf'(t) \leq f(t)$ for all $t \geq 0$;

(7) $f(t)/\sqrt{t} \to a > 0$ as $t \to +\infty$.

(8) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1 \\ C|t|^{1/2}, & |t| \geq 1. \end{cases}$$

(9) $|f(t)f'(t)| \leq 1/2^{(p-1)/p}$ *for all* $t \in \mathbb{R}$.

Proof. See [28]. ∎

The next lemma can be found in [1], however for convenience of the reader we will write its proof.

Lemma 2.2 *Let* (v_n) *be a sequence in* $W^{1,p}(\mathbb{R}^N)$ *verifying* $\displaystyle\int_{\mathbb{R}^N} |f(v_n)|^p \to 0$ *as* $n \to \infty$*. Then,*

$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p\right\} \to 0 \ \text{ as } \ n \to \infty.$$

Proof. Hereafter, once that f is odd, we can assume without loss of generality that $v_n \geq 0$ for all $n \in \mathbb{N}$. Since $f(t)/t$ is nonincreasing for $t > 0$, for each $\xi > 1$,

$$\frac{1}{\xi} + \frac{1}{\xi}\int_{\mathbb{R}^N} |f(\xi v_n)|^p \leq \frac{1}{\xi} + \xi^{p-1}\int_{\mathbb{R}^N} |f(v_n)|^p.$$

Hence, for each $\delta > 0$, fixing ξ_* sufficiently large such that $\frac{1}{\xi_*} < \frac{\delta}{2}$, we get

$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p\right\} \leq \frac{\delta}{2} + \xi_*^{p-1}\int_{\mathbb{R}^N} |f(v_n)|^p.$$

Thus,

$$\limsup_{n\to\infty}\left(\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p\right\}\right) \leq \frac{\delta}{2} \quad \text{for all } \delta > 0$$

which proves the proposition. ∎

Repeating the same type of arguments explored in the proof of the last lemma, we have the following result

Corollary 2.1 *Let (v_n) be a sequence in $X_{\epsilon,\lambda}$ with*

$$\int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \to 0 \text{ as } n \to +\infty.$$

Then

$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(\xi v_n)|^p\right\} \to 0 \; n \to +\infty.$$

The next lemma is related to a claim made in [1], which wasn't proved in that paper. Here, we decide to show its proof.

Lemma 2.3 *The function*

$$\|v\| = |\nabla v|_p + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} |f(\xi v)|^p\right].$$

is a norm in $W^{1,p}(\mathbb{R}^N)$. Moreover, $\| \;\|$ is equivalent to the usual norm in $W^{1,p}(\mathbb{R}^N)$.

Proof. We will omit the proof that $\| \;\|$ is a norm, because we can repeat with few modifications, the same arguments used by Severo [27]. From the hypotheses on f,

$$0 \le |f(t)| \le |t| \;\; \forall t \in \mathbb{R}^N,$$

this way

$$\int_{\mathbb{R}^N} |f(\xi v)|^p \le \xi^p \int_{\mathbb{R}^N} |v|^p \;\; \forall \xi \ge 0,$$

from where it follows that

$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} |f(\xi v)|^p\right\} \le \inf_{\xi>0}\left\{\frac{1}{\xi} + L\xi^{p-1}\right\}$$

where

$$L = \int_{\mathbb{R}^N} |v|^p.$$

Now, let us consider the function

$$g(\xi) = \frac{1}{\xi} + L\xi^{p-1} \quad \text{for} \quad \xi > 0.$$

A direct computation implies that g has a global minimum at some $\xi_0 > 0$, which satisfies

$$g'(\xi_0) = 0 \Leftrightarrow -{\xi_0}^2 + (p-1)L{\xi_0}^{p-2} = 0.$$

Then,

$$\xi_0 = \left(\frac{1}{(p-1)L}\right)^{\frac{1}{p}}$$

and so,

$$g(\xi_0) = (L(p-1))^{\frac{1}{p}} + L\left(\frac{1}{(p-1)L}\right)^{\frac{p-1}{p}} = CL^{\frac{1}{p}}$$

for some $C > 0$. Using these informations, it follows that

$$\|v\| \leq |\nabla v|_p + C|v|_p \ \ \forall v \in W^{1,p}(\mathbb{R}^N).$$

Hence, there is $c_1 > 0$ such that

$$\|v\| \leq c_1\|v\|_{1,p} \ \ \forall v \in W^{1,p}(\mathbb{R}^N).$$

where

$$\|v\|_{1,p} = \left[\int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} |v|^p\right]^{\frac{1}{p}} \ \ \forall v \in W^{1,p}(\mathbb{R}^N).$$

Since $(W^{1,p}(\mathbb{R}^N), \| \ \ \|)$ and $(W^{1,p}(\mathbb{R}^N), \| \ \ \|_{1,p})$ are Banach spaces, the last inequality together with Closed Graphic Theorem yields $\| \ \ \|$ and $\| \ \ \|_{1,p}$ are equivalent norms. ■

Lemma 2.4 *Let (v_n) be a sequence in $W^{1,p}(\mathbb{R}^N)$ and set*

$$\mathcal{Q}(v) := \int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} |f(v)|^p,$$

and

$$\|v\| = |\nabla v|_p + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} |f(\xi v)|^p\right].$$

Then, $\mathcal{Q}(v_n) \to 0$ if, and only if, $\|v_n\| \to 0$. Moreover, (v_n) is bounded in $(W^{1,p}(\mathbb{R}^N), \| \ \ \|)$ if, and only if, $(\mathcal{Q}(v_n))$ is bounded in $\mathbb{R}$.

Proof. The first part of the lemma is an immediate consequence of Lemma 2.2, this way we will prove only the second part of the lemma.

A straightforward computation gives

$$\|v\| \leq \mathcal{Q}(v) \;\; \forall v \in W^{1,p}(\mathbb{R}^N),$$

from where it follows that if $(\mathcal{Q}(v_n))$ is bounded, then (v_n) is also bounded. On the other hand, by Lemma 2.3, (v_n) is a bounded sequence in $(W^{1,p}(\mathbb{R}^N), \|\;\;\|)$ if, and only if, (v_n) is bounded $(W^{1,p}(\mathbb{R}^N), \|\;\;\|_{1,p})$, where $\|\;\;\|_{1,p}$ is the usual norm in $W^{1,p}(\mathbb{R}^N)$. Hence, there is $M > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^p \leq M \;\text{ and }\; \int_{\mathbb{R}^N} |v_n|^p \;\; \forall n \in \mathbb{N}.$$

Recalling that

$$|f(t)| \leq |t| \;\; \forall t \geq 0,$$

we have the estimate

$$\int_{\mathbb{R}^N} |f(v_n)|^p \leq \int_{\mathbb{R}^N} |v_n|^p \leq M \;\; \forall n \in \mathbb{N},$$

which shows that $(\mathcal{Q}(v_n))$ is bounded. ■

Lemma 2.5 *The function $|f|^p$ is a convex function, and so,*

$$(|f(t)|^{p-2}f(t)f'(t) - |f(s)|^{p-2}f(s)f'(s))(t-s) \geq 0 \quad \forall t,s \in \mathbb{R}.$$

Proof. A direct computation shows that second derivative of the function

$$Q(t) = |f(t)|^p \;\text{ for } t \in \mathbb{R}$$

satisfies the equality

$$Q''(t) = \frac{p|f(t)|^{p-2}|f'(t)|^2\left((p-1) + (p-2)2^{p-1}|f(t)|^p\right)}{1 + 2^{p-1}|f(t)|^p} > 0 \;\; \forall t \in \mathbb{R} \setminus \{0\},$$

implying that Q is a convex function. From this,

$$(Q'(t) - Q'(s))(t-s) \geq 0 \;\; \forall t,s \in \mathbb{R}$$

that is,

$$(|f(t)|^{p-2}f(t)f'(t) - |f(s)|^{p-2}f(s)f'(s))(t-s) \geq 0 \quad \forall t,s \in \mathbb{R},$$

finishing the proof. ■

Lemma 2.6 *Let $(v_n) \subset W^{1,p}(\mathbb{R}^N)$ be a sequence of nonnegative functions such that $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e in $\mathbb{R}^N$ and*

$$\int_{\mathbb{R}^N} (|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v))(v_n - v) \to 0 \;\; as \;\; n \to +\infty.$$

Then,

$$\int_{\mathbb{R}^N} |f(v_n - v)|^p \to 0 \;\; as \;\; n \to +\infty.$$

Proof. By hypothesis,

$$\int_{\mathbb{R}^N} (|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v))(v_n - v) = o_n(1)$$

or equivalently,

$$\int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n = \int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v +$$

$$\int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v)(v_n - v) + o_n(1).$$

Once that $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v)(v_n - v) = o_n(1)$$

and so,

$$\int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n = \int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v + o_n(1).$$

Recalling that

$$|f(t)| \leq |t| \;\; \text{and} \;\; |f'(t)| \leq 1 \;\; \forall t \in \mathbb{R},$$

it follows that $(|f(v_n)|^{p-2} f(v_n) f'(v_n))$ is bounded sequence in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$. Hence,

$$\int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v \to \int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v) v$$

which gives

$$\int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \to \int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v) v.$$

From Lemma 2.1,

$$|f(t)|^p \leq 2|f(t)|^{p-2} f(t) f'(t) t \quad \forall t \geq 0$$

then,

$$|f(v_n)|^p \leq |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \quad \forall n \in \mathbb{N}.$$

Using the above informations together with Lebesgue's Theorem, we deduce

$$\int_{\mathbb{R}} |f(v_n)|^p \to \int_{\mathbb{R}} |f(v)|^p.$$

On the other hand, since $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$, we have the inequality

$$|f(v_n - v)| = f(|v_n - v|) \leq f(v_n + v) \leq f(v_n) + v \ \ \forall n \in \mathbb{N}$$

which gives

$$|f(v_n - v)|^p \leq 2^p (|f(v_n)|^p + |v|^p) \ \ \forall n \in \mathbb{N}.$$

Combining the last inequality with Lebesgue's Theorem, we get

$$\int_{\mathbb{R}^N} |f(v_n - v)|^p \to 0,$$

concluding the proof of the lemma. ■

Corollary 2.2 *Let $(v_n) \subset W^{1,p}(\mathbb{R}^N)$ be a sequence of nonnegative functions such that $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e in $\mathbb{R}^N$ and the below limits hold*

$$\int_{\mathbb{R}^N} (|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v))(v_n - v) \to 0 \ \ \textit{as} \ \ n \to +\infty \tag{2.3}$$

and

$$\int_{\mathbb{R}^N} \left\langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v, \nabla v_n - \nabla v \right\rangle \to 0 \ \ \textit{as} \ \ n \to +\infty. \tag{2.4}$$

Then, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$.

Proof. By Lemma 2.6, the limit (2.3) leads to

$$\int_{\mathbb{R}^N} |f(v_n - v)|^p \to 0 \text{ as } n \to +\infty.$$

On the other hand, the limit (2.4) implies that

$$\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \to 0 \text{ as } n \to +\infty.$$

The above limits give

$$\mathcal{Q}(v_n - v) \to 0 \text{ as } n \to +\infty,$$

and so, by Lemma 2.4

$$\|v_n - v\| \to 0 \text{ as } n \to +\infty$$

or equivalently,

$$v_n \to v \text{ in } W^{1,p}(\mathbb{R}^N),$$

proving the lemma. ■

3 The Palais-Smale condition

In this Section, the main goal is to show that $I_{\epsilon,\lambda}$ satisfies the Palais-Smale condition. To this end, we have to prove some technical lemmas.

Lemma 3.1 *Suppose that h satisfies $(H_1) - (H_3)$. Let $(v_n) \subset X_{\epsilon,\lambda}$ be a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$. Then there exists a constant $K > 0$, independent of ϵ and λ, such that*

$$\limsup_{n\to\infty} \|v_n\|_{\epsilon,\lambda} \le K$$

for all $\epsilon, \lambda > 0$

Proof. Using (H_3) and Lemma 2.1(6),

$$c + o_n(1)\|v_n\|_{\epsilon,\lambda} \ge (\frac{1}{p} - \frac{1}{\theta}) \int_{\mathbb{R}^N} |\nabla v_n|^p + (\frac{1}{p} - \frac{1}{2\theta}) \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p,$$

where $o_n(1) \to 0$ as $n \to \infty$. Recalling that $|\nabla v_n|_p \leq 1 + |\nabla v_n|_p^p$,

$$c + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq \frac{(\theta - p)}{p\theta}\Big(|\nabla v_n|_p - 1 + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p\Big) \quad (3.1)$$

from where it follows the inequality

$$\begin{aligned} c_1 + o_n(1)\|v_n\|_{\epsilon,\lambda} &\geq \frac{(\theta - p)}{p\theta}\Big(|\nabla v_n|_p + 1 + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p\Big) \\ &\geq \frac{(\theta - p)}{p\theta}\|v_n\|_{\epsilon,\lambda}. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \|v_n\|_{\epsilon,\lambda} \leq c_1 \frac{p\theta}{(\theta - p)} := K.$$

■

Lemma 3.2 *Suppose that h satisfies $(H_1) - (H_3)$. Let $(v_n) \subset X_{\epsilon,\lambda}$ be a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$. Then $c \geq 0$, and if $c = 0$, we have that $v_n \to 0$ in $X_{\epsilon,\lambda}$.*

Proof As in the proof of Lemma 3.1,

$$c + o_n(1)\|v_n\|_{\epsilon,\lambda} = I_{\epsilon,\lambda}(v_n) - \frac{1}{\theta} I'_{\epsilon,\lambda}(v_n)v_n \geq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|v_n\|_{\epsilon,\lambda} \geq 0 \quad (3.2)$$

that is

$$c + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq 0.$$

The boundedness of (v_n) in $X_{\epsilon,\lambda}$ gives $c \geq 0$ after passage to the limit as $n \to \infty$. If $c = 0$, the inequality (3.2) gives $v_n \to 0$ in $X_{\epsilon,\lambda}$ as $n \to \infty$, finishing the proof of Lemma 3.2. ■

Lemma 3.3 *Suppose that h satisfies $(H_1) - (H_3)$. Let $c > 0$ and (v_n) be a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$. Then, there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |f(v_n)|^q \geq \delta,$$

with δ being independent of λ and ϵ.

Proof From $(H_1)-(H_2)$, there exists a constant $C>0$ such that

$$|h(t)t| \leq \frac{1}{4}|t|^p + C|t|^q \ \ \forall t \in \mathbb{R}. \tag{3.3}$$

Now, $I'_{\epsilon,\lambda}(v_n)v_n = o_n(1)$ and Lemma 2.1(6) give

$$\frac{1}{2}\left[\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(v_n)|^p\right] \leq \int_{\mathbb{R}^N} h(f(v_n)f(v_n). \tag{3.4}$$

Combining (3.3) with (3.4),

$$\frac{1}{4}\left[\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(v_n)|^p\right] \leq C\int_{\mathbb{R}^N} |f(v_n)|^q. \tag{3.5}$$

On the other hand, we have the equality

$$\frac{1}{p}\left[\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(v_n)|^p\right] = I_{\epsilon,\lambda}(v_n) + \int_{\mathbb{R}^N} H(f(v_n))$$

which combined with (H_3) and $I_{\epsilon,\lambda}(v_n) = c + o_n(1)$ leads to

$$\liminf_{n\to\infty}\left[\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x)+1)|f(v_n)|^p\right] \geq pc > 0. \tag{3.6}$$

Now, the lemma follows from (3.5) and (3.6). ■

Lemma 3.4 *Suppose that h satisfies $(H_1)-(H_3)$ and A satisfies $(A_1)-(A_2)$. Let $d>0$ be an arbitrary number. Given any $\epsilon>0$ and $\eta>0$, there exist $\Lambda_\eta>0$ and $R_\eta>0$, which are independent of ϵ, such that if (v_n) is a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$ with $c \leq d$ and $\lambda \geq \Lambda_\eta$, then*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{R_\eta}(0)} |f(v_n)|^q < \eta.$$

Proof. Given any $R>0$, define

$$X(R) = \{x \in \mathbb{R}^N : |x| > R;\ A(\epsilon x) \geq K_0\}$$

and

$$Y(R) = \{x \in \mathbb{R}^N : |x| > R;\ A(\epsilon x) < K_0\}.$$

Observe that

$$\int_{X(R)} |f(v_n)|^p \leq \frac{1}{\lambda K_0 + 1} \int_{X(R)} (\lambda A(\epsilon x) + 1)|f(v_n)|^p.$$

From Lemma 3.1, there exists $K > 0$ such that

$$\limsup_{n\to\infty} \int_{X(R)} |f(v_n)|^p \leq \frac{K}{\lambda K_0 + 1}. \tag{3.7}$$

On the other hand, by Hölder inequality

$$\int_{Y(R)} |f(v_n)|^p \leq \Big(\int_{Y(R)} |f(v_n)|^{p^*} \Big)^{\frac{p}{p^*}} (\mu(Y(R)))^{\frac{p}{N}}.$$

Using Sobolev embeddings together with Lemmas 2.1 and 3.1, there exists a constant $\widehat{K} > 0$ such that

$$\limsup_{n\to\infty} \int_{Y(R)} |f(v_n)|^p \leq \widehat{K}(\mu(Y(R)))^{\frac{p}{N}}, \tag{3.8}$$

where the constant $\widehat{K}$ is uniform on $c \in [0, d]$. Since

$$Y(R) \subset \{x \in \mathbb{R}^N : A(\epsilon x) \leq K_0\}$$

it follows from (A_2)

$$\lim_{R\to\infty} \mu(Y(R)) = 0. \tag{3.9}$$

Using interpolation,

$$|f(v_n)|_{L^q(\mathbb{R}^N\setminus B_R(0))} \leq |f(v_n)|^{\alpha}_{L^p(\mathbb{R}^N\setminus B_R(0))} |f(v_n)|^{1-\alpha}_{L^{p^*}(\mathbb{R}^N\setminus B_R(0))}$$

for some $\alpha \in (0, 1)$. Then, by Lemma 3.1, there exists a constant $\widetilde{K} > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |f(v_n)|^q \leq \widetilde{K} \limsup_{n\to\infty} \Big(\int_{\mathbb{R}^N\setminus B_R(0)} |f(v_n)|^p \Big)^{\frac{q\alpha}{p}}. \tag{3.10}$$

Combining (3.7) with (3.8) and (3.9), given $\eta > 0$, we can fix $R = R_\eta$ and $\Lambda_\eta > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |f(v_n)|^p \leq \Big(\frac{\eta}{2\widetilde{K}}\Big)^{\frac{p}{q\alpha}} \tag{3.11}$$

for all $\lambda \geq \Lambda_\eta$. Consequently, from (3.10) and (3.11),

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |f(v_n)|^q \leq \eta.$$

concluding the proof of the lemma. ■

As a first consequence of the last lemma, we have the following result

Corollary 3.1 *If (v_n) is a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$ and λ is large enough, then its weak limit is nontrivial provided that $c > 0$.*

The next result shows that $I_{\epsilon,\lambda}$ satisfies the Palais-Smale condition for λ sufficiently large for ϵ arbitrary.

Proposition 3.1 *Suppose that $(H_1)-(H_3)$ and $(A_1)-(A_2)$ hold. Then for any $d > 0$ and $\epsilon > 0$ there exists $\Lambda > 0$, independent of ϵ, such that $I_{\epsilon,\lambda}$ satisfies the $(PS)_c$ condition for all $c \leq d, \lambda \geq \Lambda$ and $\epsilon > 0$. That is, any sequence $(v_n) \subset X_{\epsilon,\lambda}$ satisfying*

$$I_{\epsilon,\lambda}(v_n) \to c \ \textit{ and } \ I'_{\epsilon,\lambda}(v_n) \to 0, \tag{3.12}$$

for $c \leq d$, has a strongly convergent subsequence in $X_{\epsilon,\lambda}$.

Proof Given any $d > 0$ and $\epsilon > 0$, take $c \leq d$ and let (v_n) be a $(PS)_c$ sequence for $I_{\epsilon,\lambda}$. From Lemma 3.1, there are a subsequence still denoted by (v_n) and $v \in X_{\epsilon,\lambda}$ such that (v_n) is weakly convergent to v in $X_{\epsilon,\lambda}$. If $\widetilde{v}_n = v_n - v$, arguing as in [1, Lemma 3.7], it follows that

$$I_{\epsilon,\lambda}(\widetilde{v}_n) = I_{\epsilon,\lambda}(v_n) - I_{\epsilon,\lambda}(v) + o_n(1) \tag{3.13}$$

and

$$I'_{\epsilon,\lambda}(\widetilde{v}_n) \to 0. \tag{3.14}$$

Once that $I'_{\epsilon,\lambda}(v) = 0$, (H_3) gives

$$I_{\epsilon,\lambda}(v) = I_{\epsilon,\lambda}(v) - \frac{1}{\theta} I'_{\epsilon,\lambda}(v)v \geq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|v\|_{\epsilon,\lambda,} \geq 0. \tag{3.15}$$

Setting $c' = c - I_{\epsilon,\lambda}(v)$, by (3.13)-(3.15), we deduce that $c' \leq d$ and $(\widetilde{v}_n)$ is a $(PS)_{c'}$ sequence for $I_{\epsilon,\lambda,}$, thus by Lemma 3.2, we have $c' \geq 0$. We claim

that $c' = 0$. On the contrary, suppose that $c' > 0$. From Lemma 3.3, there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |f(\widetilde{v}_n)|^q > \delta. \tag{3.16}$$

Letting $\eta = \frac{\delta}{2}$ and applying Lemma 3.4, we get $\Lambda > 0$ and $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(\widetilde{v}_n)|^q < \frac{\delta}{2} \tag{3.17}$$

for the corresponding $(PS)_{c'}$ sequence for $I_{\epsilon,\lambda}$ for all $\lambda \geq \Lambda$. Combining (3.16) with (3.17) and using the fact that $\widetilde{v}_n \rightharpoonup 0$ in $X_{\epsilon,\lambda}$, we derive

$$\delta \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |f(\widetilde{v}_n)|^q \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(\widetilde{v}_n)|^q \leq \frac{\delta}{2}$$

which is impossible, then $c' = 0$. Thereby, by Lemma 3.2, $\widetilde{v}_n \to 0$ in $X_{\epsilon,\lambda}$, that is, $v_n \to v$ in $X_{\epsilon,\lambda}$ and the proof of Proposition 3.1 is complete. ■

In closing this section, we proceed with the study of $(PS)_{c,\infty}$ sequences, that is, sequences (v_n) in $X_{\epsilon,\lambda}$ verifying:

i) $\lambda_n \to \infty$

ii) $(I_{\epsilon,\lambda_n}(v_n))$ is bounded

iii) $\|I'_{\epsilon,\lambda_n}(v_n)\|^*_{\epsilon,\lambda_n} \to 0$

where $\| \ \|^*_{\epsilon,\lambda_n}$ is defined by

$$\|\varphi\|^*_{\epsilon,\lambda_n} = \sup\{|\varphi(u)|;\ u \in X_{\epsilon,\lambda_n},\ \|u\|_{\epsilon,\lambda} \leq 1\} \ \text{ for } \ \varphi \in X^*_{\epsilon,\lambda_n}.$$

Proposition 3.2 *Suppose that $(H_1) - (H_3)$ and $(A_1) - (A_2)$ hold. Assume that $(v_n) \subset W^{1,p}(\mathbb{R}^N)$ is a $(PS)_{c,\infty}$ sequence. Then for each $\epsilon > 0$ fixed, there exists a subsequence still denoted by (v_n) and $v_\epsilon \in W^{1,p}(\mathbb{R}^N)$ such that*

i) $v_n \to v_\epsilon$ *in* $W^{1,p}(\mathbb{R}^N)$. *Moreover,* $v_\epsilon = 0$ *on* Ω^c_ϵ *and* $v_\epsilon \in W^{1,p}(\mathbb{R}^N) \bigcap L^\infty_{loc}(\mathbb{R}^N)$ *is a solution of*

$$\begin{cases} -\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), \ \ in \ \Omega_\epsilon \\ \\ v > 0 \ in \ \Omega \ \ and \ v = 0 \ \ on \ \partial\Omega_\epsilon \end{cases}$$

where $\Omega_\epsilon = \frac{\Omega}{\epsilon}$.

ii) $\lambda_n \int_{\mathbb{R}^N} |f(v_n)|^p \to 0.$

iii) $\|v_n - v\|_{\epsilon,\lambda_n} \to 0.$

Proof. As in the proof of Lemma 3.1, the sequence $(\|v_n\|_{\epsilon,\lambda_n})$ is bounded in $\mathbb{R}$. Thus, we can extract a subsequence $v_n \rightharpoonup v_\epsilon$ weakly in $X_{\epsilon,\lambda}$. For each $m \in \mathbb{N}$, we define the set

$$C_m = \Big\{x \in \mathbb{R}^N : A_\epsilon(x) \geq \frac{1}{m}\Big\}, \text{ where } A_\epsilon(x) = A(\epsilon x)$$

which satisfies

$$\int_{C_m} |f(v_n)|^p \leq m \int_{C_m} A_\epsilon(x)|f(v_n)|^p \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (1 + \lambda_n A_\epsilon(x))|f(v_n)|^p.$$

Thus, from Lemma 3.1,

$$\int_{C_m} |f(v_n)|^p \leq \frac{mK}{\lambda_n} \text{ for } n \in \mathbb{N},$$

for some constant $K > 0$. Hence by Fatou's Lemma,

$$\int_{C_m} |f(v_\epsilon)|^p = 0$$

after to passage to the limit as $n \to \infty$. Thus $f(v_\epsilon) = 0$ almost everywhere in C_m. Once that $f(t) = 0$ if, and only if $t = 0$, it follows that $v_\epsilon = 0$ almost everywhere in C_m. Observing that

$$\mathbb{R}^N \setminus A_\epsilon^{-1}(0) = \cup_{m=1}^{\infty} C_m,$$

we deduce that $v_\epsilon = 0$ almost everywhere in $\mathbb{R}^N \setminus A_\epsilon^{-1}(0)$. Now, recalling that $A_\epsilon^{-1}(0) = \overline{\Omega}_\epsilon \cup D_\epsilon$ and $\mu(D_\epsilon) = \mu(\frac{1}{\epsilon}D) = 0$, it follows that $v_\epsilon = 0$ almost everywhere in $\mathbb{R}^N \setminus \overline{\Omega}_\epsilon$. As $\partial\Omega_\epsilon$ is a smooth set, let us conclude that $v_\epsilon \in W_0^{1,p}(\Omega_\epsilon)$.

Arguing as in Lemma 3.4, we can assert that given any $\eta > 0$ there exists $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^p < \eta \tag{3.18}$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}|f(v_n)|^q<\eta. \tag{3.19}$$

From $(H_1)-(H_2)$, for each $\tau>0$ there exists $C_\tau>0$ such that

$$|h(s)|\leq\tau|s|^{p-1}+C_\tau|s|^{q-1}\ \text{ for all }\ s\in\mathbb{R}.$$

This inequality combined with Sobolev's embeddings and the limits (3.18) and (3.19) yields there is a subsequence, still denoted by (v_n), such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(f(v_n))f'(v_n)v_n=\int_{\mathbb{R}^N}h(f(v_\epsilon))f'(v_\epsilon)v_\epsilon, \tag{3.20}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(f(v_n))f'(v_\epsilon)v_\epsilon=\int_{\mathbb{R}^N}h(f(v_\epsilon))f'(v_\epsilon)v_\epsilon \tag{3.21}$$

In the sequel, define

$$P_n=\int_{\mathbb{R}^N}\left\langle|\nabla v_n|^{p-2}\nabla v_n-|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\nabla v_n-\nabla v_\epsilon\right\rangle+$$

$$+\int_{\mathbb{R}^N}(|f(v_n)|^{p-2}f(v_n)f'(v_n)-|f(v_\epsilon)|^{p-2}f(v_\epsilon)f'(v_\epsilon))(v_n-v_\epsilon)$$

and observe that

$$P_n\leq\int_{\mathbb{R}^N}\left\langle|\nabla v_n|^{p-2}\nabla v_n-|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\nabla v_n-\nabla v_\epsilon\right\rangle+$$

$$\int_{\mathbb{R}^N}(\lambda_nA_\epsilon(x)+1)(|f(v_n)|^{p-2}f(v_n)f'(v_n)-|f(v_\epsilon)|^{p-2}f(v_\epsilon)f'(v_\epsilon))(v_n-v_\epsilon)=$$

$$I'_{\epsilon,\lambda_n,}(v_n)v_n-I'_{\epsilon,\lambda_n}(v_n)v_\epsilon+\int_{\mathbb{R}^N}h(f(v_n))f(|v_n)|^{p-2}f(v_n)f'(v_n)v_n$$
$$-\int_{\mathbb{R}^N}h(f(v_n))f(|v_n)|^{p-2}f(v_n)f'(v_n)v_\epsilon+o_n(1).$$

Thus, by (3.20) and (3.21) it follows that $P_n=o_n(1)$, that is,

$$\int_{\mathbb{R}^N}\left\langle|\nabla v_n|^{p-2}\nabla v_n-|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\nabla v_n-\nabla v_\epsilon\right\rangle=o_n(1) \tag{3.22}$$

and

$$\int_{\mathbb{R}^N}(|f(v_n)|^{p-2}f(v_n)f'(v_n)-|f(v_\epsilon)|^{p-2}f(v_\epsilon)f'(v_\epsilon))(v_n-v_\epsilon)=o_n(1). \quad (3.23)$$

The limits (3.23) and (3.22) combined with Corollary 2.2 give

$$v_n \to v_\epsilon \text{ strongly in } W^{1,p}(\mathbb{R}^N). \quad (3.24)$$

Now, using the fact that $(\lambda_n A_\epsilon(x)+1)v_\epsilon(x)=v_\epsilon(x)$ a.e in $\mathbb{R}^N$ and that for each $\phi \in C_0^\infty(\Omega_\epsilon)$, $I'_{\epsilon,\lambda_n}(v_n)\phi = o_n(1)$, we have that

$$\int_{\mathbb{R}^N}(|\nabla v_n|^{p-2}\nabla v_n\nabla\phi+|f(v_n)|^{p-2}f(v_n)f'(v_n)\phi)=\int_{\mathbb{R}^N}h(f(v_n))f'(v_n)\phi+o_n(1).$$

This together with (3.24) yields

$$\int_{\Omega_\epsilon}(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\nabla\phi+|f(v_\epsilon)|^{p-2}f(v_\epsilon)f'(v_\epsilon)\phi)=\int_{\Omega_\epsilon}h(f(v_\epsilon))f'(v_\epsilon)\phi$$

and hence

$$\int_{\Omega_\epsilon}(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\nabla w+|f(v_\epsilon)|^{p-2}f(v_\epsilon)f'(v_\epsilon)w)=\int_{\Omega_\epsilon}h(f(v_\epsilon))f'(v_\epsilon)w, \quad (3.25)$$

for all $w \in W_0^{1,p}(\Omega_\epsilon)$. Arguing as [1, Proposition 3.6], we can prove that $v_\epsilon \in L^\infty(\mathbb{R}^N)$. Thereby, v_ϵ is a solution of

$$\begin{cases} -\Delta_p v+|f(v)|^{p-2}f(v)f'(v)=h(f(v))f'(v), \text{ in } \Omega_\epsilon \\ \\ v>0 \text{ in } \Omega \text{ and } u=0 \text{ on } \partial\Omega_\epsilon \end{cases}$$

and the proof of $i)$ is complete.

To deduce $ii)$, we start observing that

$$\begin{aligned} &\int_{\mathbb{R}^N}|\nabla v_n|^p+\int_{\mathbb{R}^N}|f(v_n)|^{p-2}f(v_n)f'(v_n)v_n \\ +\ &\lambda_n\int_{\mathbb{R}^N}A_\epsilon|f(v_n)|^{p-2}f(v_n)f'(v_n)v_n=\int_{\mathbb{R}^N}h(f(v_n))f'(v_n)v_n+o_n(1). \end{aligned}$$

The last equality, combined with (3.24) and (3.25) leads to

$$\lim_{n\to\infty}\lambda_n\int_{\mathbb{R}^N}A_\epsilon(x)|f(v_n)|^{p-2}f(v_n)f'(v_n)v_n=0.$$

This limit together with Lemma 2.1(6) implies that

$$\lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n)|^p = 0,$$

proving ii).

For to prove iii), we observe that

$$\lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n - v_\epsilon)|^p = \lambda_n \int_{\mathbb{R}^N\setminus\Omega} A_\epsilon(x)|f(v_n)|^p \leq \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n)|^p$$

because $v_\epsilon = 0$ in Ω. Hence,

$$\lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n - v_\epsilon)|^p \to 0 \text{ as } n \to +\infty$$

which together with Corollary 2.2 leads to

$$\int_{\mathbb{R}^N} (1 + \lambda_n A_\epsilon(x))|f(v_n - v_\epsilon)|^p \to 0 \text{ as } n \to +\infty. \tag{3.26}$$

Recalling that

$$\|v_n - v_\epsilon\|_{\epsilon,\lambda_n} \leq |\nabla v_n - \nabla v_\epsilon|_p + \int_{\mathbb{R}^N} (1 + \lambda_n A_\epsilon(x))|f(v_n - v_\epsilon)|^p,$$

it follows from (3.24) and (3.26),

$$\lim_{n\to\infty} \|v_n - v_\epsilon\|_{\epsilon,\lambda_n} = 0,$$

which proves *iii*), and the proof of Proposition 3.2 is complete. ■

Corollary 3.2 *Suppose that $(A_1) - (A_2)$ and $(H_1) - (H_4)$ hold. Then for each $\epsilon > 0$ and a sequence (v_n) of solutions of (P_{ϵ,λ_n}) with $\lambda_n \to \infty$ and $\limsup_{n\to\infty} I_{\epsilon,\lambda_n}(v_n) < \infty$, there exists a subsequence that converges strongly in $W^{1,p}(\mathbb{R}^N)$ to a solution of the problem*

$$\begin{cases} -\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), & \text{in } \Omega_\epsilon \\ \\ v > 0 \text{ in } \Omega_\epsilon \quad \text{and } v = 0 \ \ \partial\Omega_\epsilon. \end{cases}$$

Proof. By assumptions, there exist $c \in \mathbb{R}$ and a subsequence of (v_n), still denoted by (v_n), such that (v_n) is a $(PS)_{c,\infty}$ sequence. The rest of the proof follows from Proposition 3.2. ■

4 Behavior of minimax levels

This section is devoted to the study of the behavior of the minimax levels with respect to parameter λ and ϵ. For this purpose, we introduce some notations. In the next, $\mathcal{M}_{\epsilon,\lambda}$ denotes the Nehari manifold associated to $I_{\epsilon,\lambda}$, that is,

$$\mathcal{M}_{\epsilon,\lambda} = \Big\{ v \in X_{\epsilon,\lambda} : \; v \neq 0 \;\text{ and }\; I'_{\epsilon,\lambda}(v)v = 0 \Big\}$$

and

$$c_{\epsilon,\lambda} = \inf_{v \in \mathcal{M}_{\epsilon,\lambda}} I_{\epsilon,\lambda}(v).$$

From $(H_1) - (H_4)$, as proved in [1, Lemma 3.3], the number $c_{\epsilon,\lambda}$ is the mountain pass minimax level associated with $I_{\epsilon,\lambda}$.

On account of the proof of Proposition 3.2, when λ is large, the following problem can be seen as a limit problem of $(D_{\epsilon,\lambda,})$ for each $\epsilon > 0$:

$$(D_\epsilon) \qquad \begin{cases} -\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), \;\text{ in }\; \Omega_\epsilon \\ \\ v > 0 \text{ in } \Omega \;\text{ and }\; v = 0 \;\text{ on }\; \partial\Omega_\epsilon \end{cases}$$

whose corresponding functional is given by

$$E_\epsilon(v) = \frac{1}{p} \int_{\Omega_\epsilon} (|\nabla v|^p + |f(v)|^p) - \int_{\Omega_\epsilon} H(f(v))$$

for every $v \in W_0^{1,p}(\Omega_\epsilon)$. Here and subsequently, $\mathcal{M}_\epsilon$ denotes the Nehari manifold associated to E_ϵ and

$$c(\epsilon, \Omega) = \inf_{v \in \mathcal{M}_\epsilon} E_\epsilon(v)$$

stands for the mountain pass minimax associated with E_ϵ. Since $0 \in \Omega$, there is $r > 0$ such that $B_r = B_r(0) \subset \Omega$ and $B_{\frac{r}{\epsilon}} = B_{\frac{r}{\epsilon}}(0) \subset \Omega_\epsilon$. We will denote by $E_{\epsilon,B_r} : W_0^{1,p}(B_{\frac{r}{\epsilon}}(0)) \to \mathbb{R}$ the functional

$$E_{\epsilon,B_r}(v) = \frac{1}{p} \int_{B_{\frac{r}{\epsilon}}} (|\nabla v|^p + |f(v)|^p) - \int_{B_{\frac{r}{\epsilon}}} H(f(v)).$$

Furthermore, we write $\mathcal{M}_{\epsilon,B_r}$ the Nehari manifold associated to E_{ϵ,B_r} and

$$c(\epsilon, B_r) = \inf_{v \in \mathcal{M}_{\epsilon,B_r}} E_{\epsilon,B_r}(v).$$

Once that $B_{\frac{r}{\epsilon}} \subset \Omega_\epsilon$, we have $c(\epsilon,\Omega) \le c(\epsilon, B_r)$ for every $\epsilon > 0$.

Here it is important the number c_∞, which denotes the mountain minimax value associated to

$$I_\infty(v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + |f(v)|^p) - \int_{\mathbb{R}^N} H(f(v)) \text{ for all } v \in W_0^{1,p}(\mathbb{R}^N),$$

whose existence is guaranteed by [2, Lemma 3.1]. Since $I_{\epsilon,\lambda}(tv) \ge I_\infty(tv)$ for all $t > 0$ and $v \in W^{1,p}(\mathbb{R}^N)$,

$$c_{\epsilon,\lambda} \ge c_\infty.$$

Proposition 4.1 *Suppose* $(H_1)-(H_4)$ *and* $(A_1)-(A_2)$ *hold. Let* $\epsilon > 0$ *be an arbitrary number. Then,*

$$\lim_{\lambda\to\infty} c_{\epsilon,\lambda} = c(\epsilon,\Omega).$$

Proof. By Proposition 3.1 and Mountain Pass Theorem, we can assume that there are two sequences, $\lambda_n \to \infty$ and $(v_n) \subset X_{\epsilon,\lambda_n}$, such that

$$I_{\epsilon,\lambda_n}(v_n) = c_{\epsilon,\lambda_n} > 0 \text{ and } I'_{\epsilon,\lambda_n}(v_n) = 0.$$

From definitions of c_{ϵ,λ_n} and $c(\epsilon,\Omega)$,

$$c_{\epsilon,\lambda_n} \le c(\epsilon,\Omega) \text{ for all } n \in \mathbb{N}$$

which implies

$$0 \le I_{\epsilon,\lambda_n}(v_n) \le c(\epsilon,\Omega) \text{ and } I'_{\epsilon,\lambda_n}(v_n) = 0.$$

Thus, for some subsequence (v_{n_j}), there exists $c \in [0, c(\epsilon,\Omega)]$ such that

$$I_{\epsilon,\lambda_{n_j}}(v_{n_j}) = c_{\epsilon,\lambda_{n_j}} \to c \text{ and } I'_{\epsilon,\lambda_{n_j}}(v_{n_j}) \to 0$$

showing that (v_{n_j}) is a $(PS)_{c,\infty}$, and so,

$$\int_{\mathbb{R}^N}|\nabla v_n|^p + \int_{\mathbb{R}^N}(\lambda_n A_\epsilon(x)+1)|f(v_n)|^p \ge pc_{\epsilon,\lambda n} \ge pc_\infty > 0 \ \forall n \in \mathbb{N}.$$

By Proposition 3.2,

$$\lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n)|^p \to 0 \text{ as } n \to +\infty$$

then,

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} |f(v_n)|^p \geq pc_\infty > 0 + o_n(1) \ \ \forall n \in \mathbb{N}, \tag{4.1}$$

implying that any subsequence of (v_n) does not converge to zero in $W^{1,p}(\mathbb{R}^N)$. From Proposition 3.2, there exist a subsequence $(v_{n_{j_k}})$ and $v \in W^{1,p}(\mathbb{R}^N)$ such that

$$v_{n_{j_k}} \to v \ \text{ strongly in } \ W^{1,p}(\mathbb{R}^N) \ \text{ and } \ v = 0 \ \text{ in } \ \mathbb{R}^N \setminus \Omega_\epsilon. \tag{4.2}$$

From (4.1) and (4.2), $v \neq 0$ in $W_0^{1,p}(\Omega_\epsilon)$ and v satisfies

$$\begin{cases} -\Delta_p u + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), \ \text{ in } \ \Omega_\epsilon \\ \\ v > 0 \text{ in } \Omega \ \text{ and } \ v = 0 \ \text{ on } \partial\Omega_\epsilon, \end{cases}$$

from where it follows that

$$E_\epsilon(v) \geq c(\epsilon, \Omega). \tag{4.3}$$

On the other hand,

$$E_\epsilon(v) = \lim_{k\to\infty} I_{\lambda_{n_{j_k}},\epsilon}(v_{n_{j_k}}) = \lim_{k\to\infty} c_{\epsilon,\lambda_{n_{j_k}}} = c \leq c(\epsilon, \Omega). \tag{4.4}$$

Therefore, (4.3) and (4.4) give

$$\lim_{k\to\infty} c_{\epsilon,\lambda_{n_{j_k}}} = c(\epsilon, \Omega).$$

As a result, $c_{\epsilon,\lambda} \to c(\epsilon, \Omega)$ as $\lambda \to \infty$, and the lemma follows. ■

Corollary 4.1 *Suppose that* $(A_1) - (A_2)$ *and* $(H_1) - (H_4)$ *hold. Then for each* $\epsilon > 0$ *and a sequence* (v_n) *of least energy solutions of* (D_{ϵ,λ_n}) *with* $\lambda_n \to \infty$ *and* $\limsup_{n\to\infty} I_{\epsilon,\lambda_n}(v_n) < \infty$*, there exists a subsequence that converges strongly in* $W^{1,p}(\mathbb{R}^N)$ *to a least energy solution of the problem*

$$\begin{cases} -\Delta_p u + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), \ \textit{ in } \ \Omega_\epsilon \\ \\ u > 0 \ \textit{in } \Omega_\epsilon \ \ \textit{and} \ \ u = 0 \ \ \partial\Omega_\epsilon. \end{cases}$$

Proof. The proof is a consequence of Propositions 3.2 and 4.1. ■

Hereafter, $r > 0$ denotes a number such that $B_r(0) \subset \Omega$ and the sets

$$\Omega_+ = \{x \in \mathbb{R}^N : \, d(x, \overline{\Omega}) \leq r\}$$

and

$$\Omega_- = \{x \in \mathbb{R}^N : \, d(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω. The existence of this r is given by condition (A_1). For each $v \in W^{1,p}(\mathbb{R}^N)$ whose positive part $v_+ = \max\{v, 0\}$ is different from zero and has a compact support, we consider the center mass of v

$$\beta(v) = \frac{\displaystyle\int_{\mathbb{R}^N} x v_+^p}{\displaystyle\int_{\mathbb{R}^N} v_+^p}.$$

Consider $R > 0$ such that $\Omega \subset B_R(0)$, thus $\Omega_\epsilon \subset B_{\frac{R}{\epsilon}}(0)$, and define the auxiliary function

$$\xi_\epsilon(t) = \begin{cases} 1, \ 0 \leq t \leq \frac{R}{\epsilon} \\ \\ \frac{R}{\epsilon t}, \quad \frac{R}{\epsilon} \leq t. \end{cases}$$

For $v \in W^{1,p}(\mathbb{R}^N), v_+ \neq 0$, define

$$\beta_\epsilon(v) = \frac{\displaystyle\int_{\mathbb{R}^N} x \xi_\epsilon(|x|) v_+^p}{\displaystyle\int_{\mathbb{R}^N} v_+^p}.$$

Now for each $y \in \mathbb{R}^N$ and $R > 2\mathrm{diam}(\Omega)$ fix

$$A_{\frac{R}{\epsilon}, \frac{r}{\epsilon}, y} = \Big\{x \in \mathbb{R}^N : \, \frac{r}{\epsilon} \leq |x - y| \leq \frac{R}{\epsilon}\Big\}.$$

We observe that if $y \notin \frac{1}{\epsilon}\Omega_+$ then $\overline{\Omega_\epsilon} \cap B_{\frac{r}{\epsilon}}(y) = \emptyset$. As a consequence

$$\overline{\Omega_\epsilon} \subset A_{\frac{R}{\epsilon}, \frac{r}{\epsilon}, y} \tag{4.5}$$

for every $y \notin \frac{1}{\epsilon}\Omega_+$. Moreover, for $y \in \mathbb{R}^N$, $\alpha(R, r, \epsilon, y)$ denotes the number

$$\alpha(R, r, \epsilon, y) = \inf\Big\{\widehat{J}_{\epsilon,y}(v) : \, \beta(v) = y \ \text{ and } \ v \in \widehat{\mathcal{M}}_{\epsilon,y}\Big\}$$

where
$$\widehat{J}_{\epsilon,y}(v) = \frac{1}{p}\int_{A_{\frac{R}{\epsilon},\frac{r}{\epsilon},y}} (|\nabla v|^p + |f(v)|^p) - \int_{A_{\frac{R}{\epsilon},\frac{r}{\epsilon},y}} H(f(v))$$
and
$$\widehat{\mathcal{M}}_{\epsilon,y} = \Big\{ v \in W_0^{1,p}(A_{\frac{R}{\epsilon},\frac{r}{\epsilon},y}) : v \neq 0 \text{ and } \widehat{J}'_{\epsilon,y}(v)v = 0 \Big\}.$$
From now on, we will write $\alpha(R,r,\epsilon,0)$ as $\alpha(R,r,\epsilon)$, $\widehat{J}_{\epsilon,0}$ as $\widehat{J}_\epsilon$ and $\widehat{\mathcal{M}}_{\epsilon,0}$ as $\widehat{\mathcal{M}}_\epsilon$.

Lemma 4.1 *Assume that $(H_1)-(H_4)$ hold. Then, there exists $\epsilon^* > 0$ such that*
$$c(\epsilon,\Omega) < \alpha(R,r,\epsilon)$$
for every $\epsilon \in (0,\epsilon^*)$.

Proof. Invoking [2, Proposition 4.1], we assert that
$$\lim_{\epsilon\to 0} \alpha(R,r,\epsilon) > c_\infty.$$
Thus, there exists $\epsilon_1 > 0$ such that
$$\alpha(R,r,\epsilon) > c_\infty + \delta \tag{4.6}$$
for all $0 < \epsilon < \epsilon_1$, for some $\delta > 0$. On the other hand, arguing as in [2, Proposition 4.2],
$$\lim_{\epsilon\to 0} c(\epsilon,B_r) = c_\infty.$$

Therefore, there exists $\epsilon_2 > 0$ such that
$$c(\epsilon,B_r) < c_\infty + \frac{\delta}{2} \text{ for all } 0 < \epsilon < \epsilon_2. \tag{4.7}$$
For $\epsilon^* = \min\{\epsilon_1,\epsilon_2\}$, (4.6) and (4.7) lead to
$$c(\epsilon,B_r) < \alpha(R,r,\epsilon)$$
for every $\epsilon \in (0,\epsilon^*)$. Now, the lemma follows of the inequality $c(\epsilon,\Omega) \leq c(\epsilon,B_r)$. ■

To conclude this section, we establish a result about the center of mass of the function in the Nehari manifold $\mathcal{M}_{\epsilon,\lambda}$.

Lemma 4.2 *Suppose* $(H_1)-(H_4)$ *and* $(A_1)-(A_2)$ *hold. Let* $\epsilon^*>0$ *given by Lemma 4.1. Then for any* $\epsilon\in(0,\epsilon^*)$, *there exists* $\lambda^*>0$ *which depends on* ϵ *such that*

$$\beta_\epsilon(v)\in\frac{1}{\epsilon}\Omega_+$$

for all $\lambda>\lambda^*, 0<\epsilon<\epsilon^*$ *and* $v\in\mathcal{M}_{\epsilon,\lambda}$ *with* $I_{\epsilon,\lambda}(v)\le c(\epsilon,B_r)$.

Proof. Suppose by contradiction that there exists a sequence (λ_n) with $\lambda_n\to\infty$ such that

$$v_n\in\mathcal{M}_{\epsilon,\lambda_n},\ I_{\epsilon,\lambda_n}(v_n)\le c(\epsilon,B_r)$$

and

$$\beta_\epsilon(v_n)\notin\frac{1}{\epsilon}\Omega_+. \tag{4.8}$$

Repeating the same arguments used in the proofs of Lemma 3.4 and Proposition 3.2, $(\|v_n\|_{\epsilon,\lambda_n})$ is a bounded sequence in $\mathbb{R}$ and there exists $v\in W^{1,p}(\mathbb{R}^N)$ such that $v_n\rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$, $v=0$ in $\mathbb{R}^N\setminus\Omega_\epsilon$ and for each $\eta>0$ there exists $R>0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}|f(v_n)|^p<\eta.$$

This fact implies that

$$f(v_n)\to f(v)\ \text{ strongly in }\ L^p(\mathbb{R}^N).$$

Hence by interpolation,

$$f(v_n)\to f(v)\ \text{ strongly in }\ L^t(\mathbb{R}^N)\ \text{ for all }\ t\in[p,p^*).$$

On the other hand, since $v_n\in\mathcal{M}_{\epsilon,\lambda_n}$, from (4.1),

$$0<pc_\infty\le\int_{\mathbb{R}^N}h(f(v_n))f'(v_n)v_n,\ \text{ for all } n\in\mathbb{N},$$

from where it follows that

$$0<pc_\infty\le\int_{\mathbb{R}^N}h(f(v))f'(v)v,$$

which yields

$$v \neq 0,\ E'_\epsilon(v)v \leq 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_\epsilon(v_n) = \beta(v). \tag{4.9}$$

From (4.8) and (4.9), $y = \beta(v) \notin \frac{1}{\epsilon}\Omega_+$, $\Omega_\epsilon \subset A_{\frac{R}{\epsilon},\frac{r}{\epsilon},y}$ and there exists $\tau \in (0,1]$ such that $\tau v \in \widehat{\mathcal{M}}_{\epsilon,y}$. Thereby,

$$\widehat{J}_{\epsilon,y}(\tau v) = E_\epsilon(\tau v) \leq \liminf_{n\to\infty} I_{\epsilon,\lambda_n,}(\tau v_n) \leq \liminf_{n\to\infty} I_{\epsilon,\lambda_n}(v_n) \leq c(\epsilon, B_r)$$

which implies

$$\alpha(R, r, \epsilon, y) \leq c(\epsilon, B_r).$$

On the other hand, since

$$\alpha(R, r, \epsilon, y) = \alpha(R, r, \epsilon)$$

we have

$$\alpha(R, r, \epsilon) \leq c(\epsilon, B_r),$$

contrary to Lemma 4.1, and the proof is complete. ■

5 Proof of Theorem 1.1

For $r > 0$ and $\epsilon > 0$, let $v_{r\epsilon} \in W_0^{1,p}(B_{\frac{r}{\epsilon}}(0))$ be a nonnegative radially symmetric function such that

$$E_{\epsilon,B_r}(v_{r\epsilon}) = c(\epsilon, B_r) \ \text{ and } \ E'_{\epsilon,B_r}(v_{r\epsilon}) = 0,$$

whose existence is proved in [2, Proposition 4.4]. For $r > 0$ and $\epsilon > 0$, define $\Psi_r : \frac{1}{\epsilon}\Omega_- \to W_0^{1,p}(\Omega_\epsilon)$ by

$$\Psi_r(y)(x) = \begin{cases} v_{r\epsilon}(|x-y|),\ x \in B_{\frac{r}{\epsilon}}(y) \\ \\ 0,\ x \notin B_{\frac{r}{\epsilon}}(y). \end{cases}$$

It is immediate that $\beta_\epsilon(\Psi_r(y)) = y$ for all $y \in \frac{1}{\epsilon}\Omega_-$. In the sequel, we denote by $I_{\epsilon,\lambda}^{c(\epsilon,B_r)}$ the set

$$I_{\epsilon,\lambda}^{c(\epsilon,B_r)} = \Big\{ v \in \mathcal{M}_{\epsilon,\lambda} :\ I_{\epsilon,\lambda}(v) \leq c(\epsilon, B_r) \Big\}.$$

We claim that

$$cat I_{\epsilon,\lambda}^{c(\epsilon,B_r)} \geq cat(\Omega) \tag{5.1}$$

for all $\epsilon \in (0,\epsilon^*)$ and $\lambda \geq \lambda^*$. In fact, suppose that

$$I_{\epsilon,\lambda}^{c(\epsilon,B_r)} = \cup_{i=1}^{n} O_i$$

where $O_i, i = 1, ..., n$, is closed and contractible in $I_{\epsilon,\lambda}^{c(\epsilon,B_r)}$, that is, there exists $h_i \in C([0,1] \times O_i, I_{\epsilon,\lambda}^{c(\epsilon,B_r)})$ such that, for every, $v \in O_i$,

$$h_i(0,v) = v \;\text{ and }\; h_i(1,u) = w_i$$

for some $w_i \in I_{\epsilon,\lambda}^{c(\epsilon,B_r)}$. Consider

$$B_i = \Psi_r^{-1}(O_i), \;\; i = 1, ..., n.$$

The sets B_i are closed and

$$\frac{1}{\epsilon}\Omega_- = B_1 \cup ... \cup B_n.$$

Consider the deformation $g_i : [0,1] \times B_i \to \frac{1}{\epsilon}\Omega_+$ given

$$g_i(t,y) = \beta_\epsilon(h_i(t, \Psi_r(y))).$$

From Lemma 4.2, the function g_i is well defined. Thus, B_i is contractile in $\frac{1}{\epsilon}\Omega_+$. Hence,

$$cat(\Omega) = cat(\Omega_\epsilon) = cat_{\frac{1}{\epsilon}\Omega_+}\big(\frac{1}{\epsilon}\Omega_-\big) \leq cat I_{\epsilon,\lambda}^{c(\epsilon,B_r)}$$

which verifies (5.1).

Now, we are ready to conclude the proof of Theorem 1.1. From Proposition 3.1 the functional $I_{\epsilon,\lambda}$ satisfies the Palais-Smale condition provided that $\lambda \geq \lambda^*$. Thus, by Lusternik-Schirelman theory, the functional $I_{\epsilon,\lambda}$ has at least $cat(\Omega)$ critical points for all $\epsilon \in (0,\epsilon^*)$ where $\epsilon^* > 0$ is given by Lemma 4.1. The proof is complete. ■

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